The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×10=10

    (i) Is $\mathbb{R}^2(\mathbb{R})$ is a subspace of $\mathbb{R}^3(\mathbb{R})$?

    (ii) Let A be a 5×4 matrix. If null space of A is a subspace of $\mathbb{R}^k$ then what is k?

    (iii) Let S be a subset of a vector space $V(F)$ and S contains zero vector of V. Then S is

    (A) linearly independent

    (B) linearly dependent

(C) Both linearly independent and linearly dependent

(D) None of the above

(*Choose the correct option*)

(iv) Write the standard basis of the vector space of polynomial in x with real co-efficient of degree ≤ 3.

(v) "The eigenvalues of a triangular matrix are the entries on its main diagonal." (*State True **or** False*)

(vi) Define inner product on $\mathbb{R}^n$.

(vii) Which vector is orthogonal to every vector in $\mathbb{R}^n$?

(viii) How do you explain *dim* $W = 1$ geometrically where W is a subspace of the vector space $\mathbb{R}^3 (\mathbb{R})$?

(ix) Let A be the 4 × 4 real matrix,

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & -2 & 0 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

Then the characteristic polynomial for A is

(A) $x^2(x-1)^2$

(B) $(x-1)^2(x+1)^2$

(C) $x^2(x+1)^2$

(D) None of the above

(Choose the correct option)

(x) What do you mean by the length of a vector in $\mathbb{R}^n$?

2. Answer the following questions :   2×5=10

(i) Let V be the vector space of all functions from the real field $\mathbb{R}$ to $\mathbb{R}$. Show that $W = \{ f : f(7) = 2 + f(1) \}$ is not a subspace of V.

(ii) Show that every subset of an independent set is independent.

(iii) Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Is v a eigenvector of A ?

(iv) Let T be the linear operator on $\mathbb{R}^3$ defined by $T(a,b,c) = (a+b, b+c, 0)$. Show that the

xy-plane $= \{(x,y,0) : x, y \in \mathbb{R}\}$ is T-invariant subspace of $\mathbb{R}^3$.

(v) Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of y onto u.

3. Answer **any four** questions :   5×4=20

(i) Prove that the non-zero vectors

$v_1, v_2, ..., v_n$ are linearly dependent if and only if one of them is a linear combination of the preceding vectors.

(ii) Let $v_1, v_2, ..., v_n$ be non-zero

eigenvectors of an operator $T : V \to V$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Prove that $v_1, v_2, ..., v_n$ are linearly independent.

(iii) Let A and B be two similar matrices of order $n \times n$. Prove that A and B have same characteristic polynomial and hence the same eigenvalues.

(iv) Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

(v) Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^2, given that $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

(vi) Define orthogonal set. If $S = \{u_1, u_2, \ldots, u_p\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then prove that S is linearly independent and hence is a basis for the subspace spanned by S.

4. (i) If a vector space V has a basis $B = \{v_1, v_2, \ldots, v_n\}$, then prove that any set in V containing more than n vectors must be linearly dependent. Also show that every basis of V must consist of exactly n vectors. 5+5=10

OR

Let U and V be vector spaces over the same field. Let $\{u_1, u_2, \ldots, u_n\}$ be a basis of U and let $v_1, v_2, \ldots, v_n$ be any arbitrary vectors in V. Prove that there exists a unique linear mapping $f : U \to V$ such that

$$f(u_1) = v_1, \ f(u_2) = v_2, \ldots, f(u_n) = v_n \qquad 10$$

(ii) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. 10

OR

State Cayley-Hamilton theorem for matrices. Use it to express $2A^5 - 3A^4 - A^2 - 4I$ as a linear polynomial in A, when $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. 10

(iii) Let T be the linear operator on $\mathbb{R}^3$, defined by

$$T(x, y, z) = (2y + z, \ x - 4y, \ 3x)$$

(a) Find the matrix of T in the basis

$$\{e_1 = (1,1,1), \ e_2 = (1,1,0), \ e_3 = (1,0,0)\}$$

(b) Verify that $[T]_e [v]_e = [T(v)]_e$ for any vector $v \in \mathbb{R}^3$.   4+6=10

OR

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigen-vectors.   10

(iv) Define orthonormal set and orthonormal basis in $\mathbb{R}^n$. Show that $\{u_1, u_2, u_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where

$$u_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

$$1+1+8=10$$

OR

Define inner product space. Show that the following is an inner product in $\mathbb{R}^2$:

$$\langle u, v \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3 x_2 y_2$$

where $u = (x_1, x_2)$, $v = (y_1, y_2)$.

Also show that for all u, v in $\mathbb{R}^2$

$$\|u + v\| \le \|u\| + \|v\| \qquad 2+5+3=10$$

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